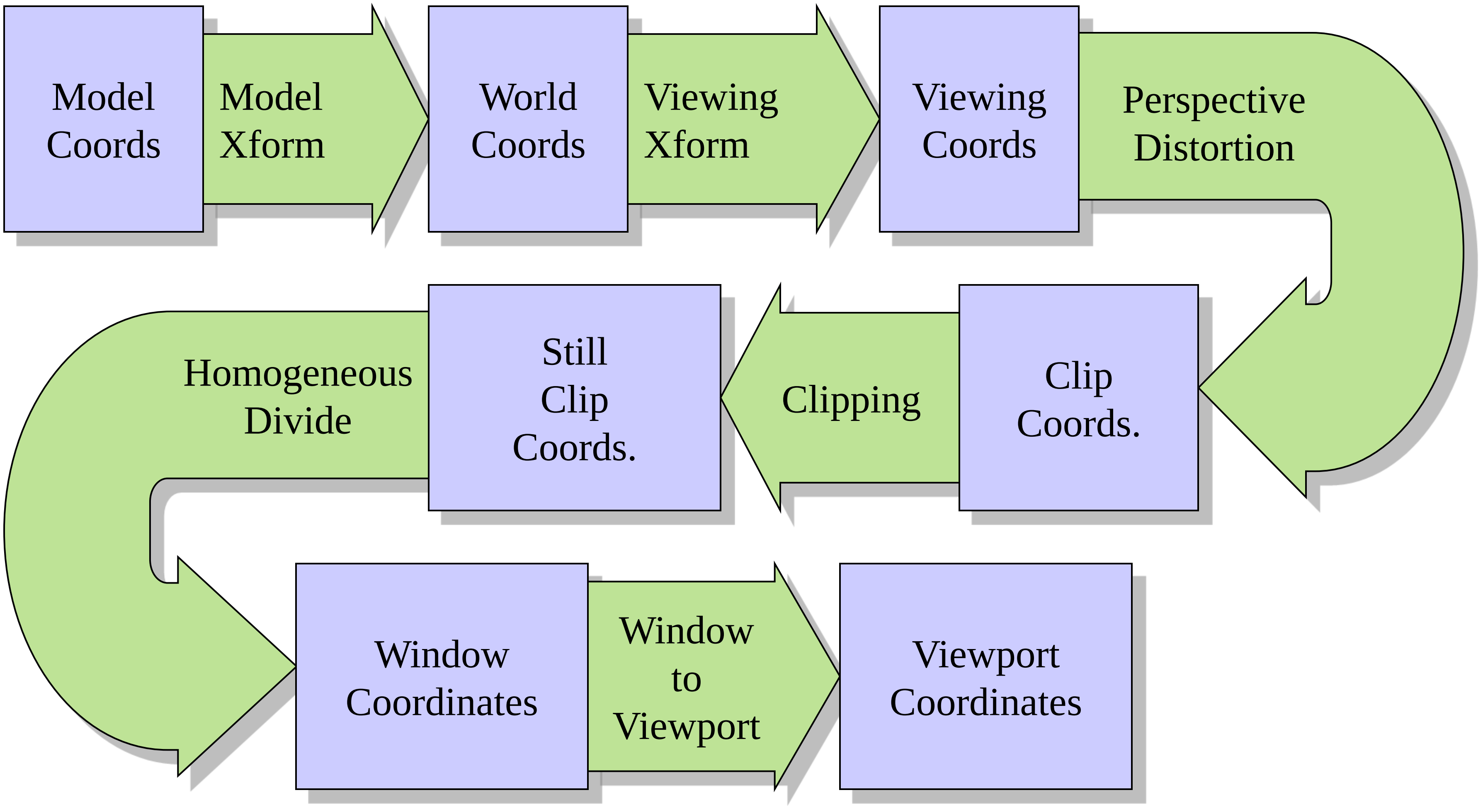
Transformational Geometry

CS418 Computer Graphics

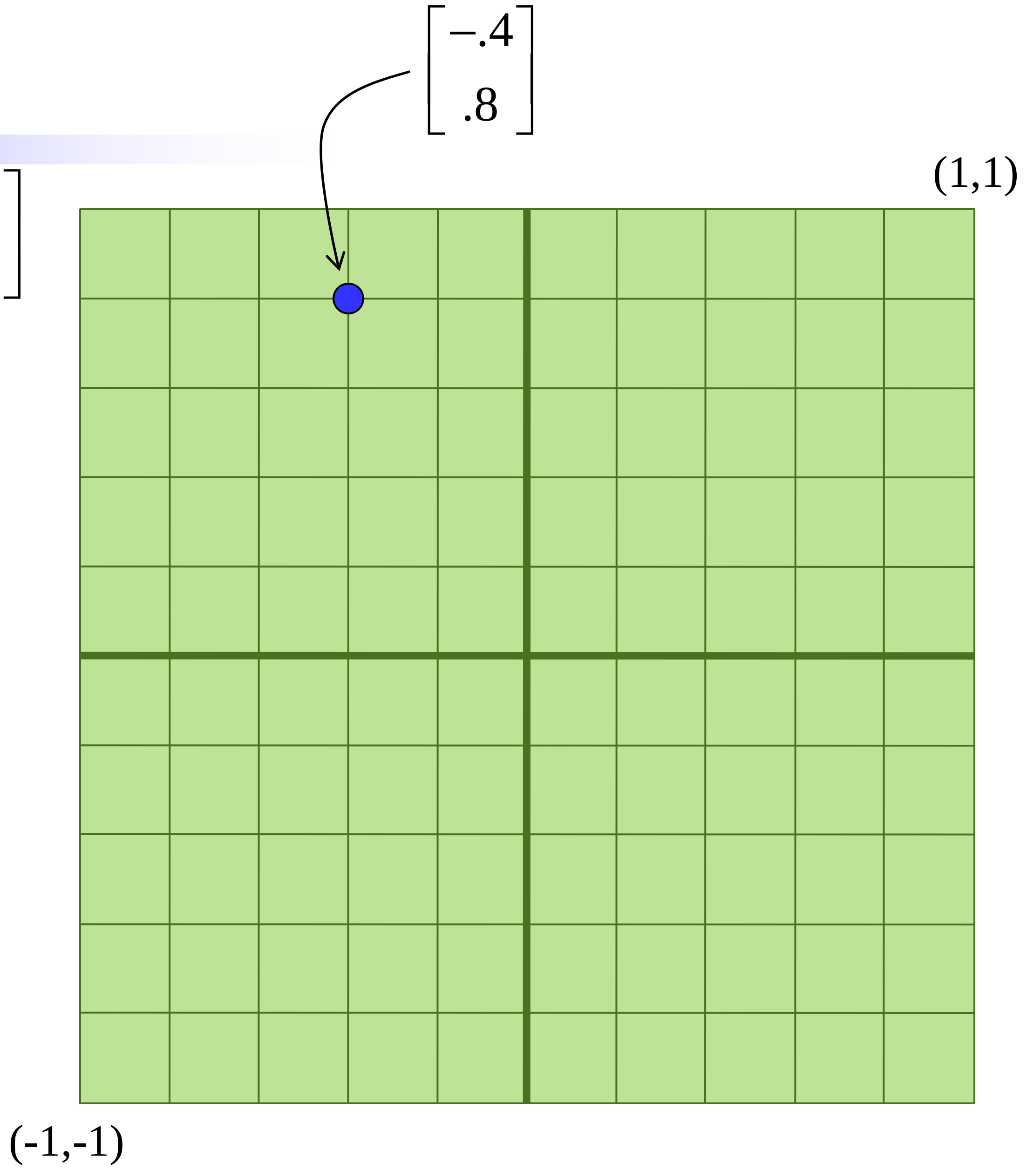
John C. Hart

Graphics Pipeline



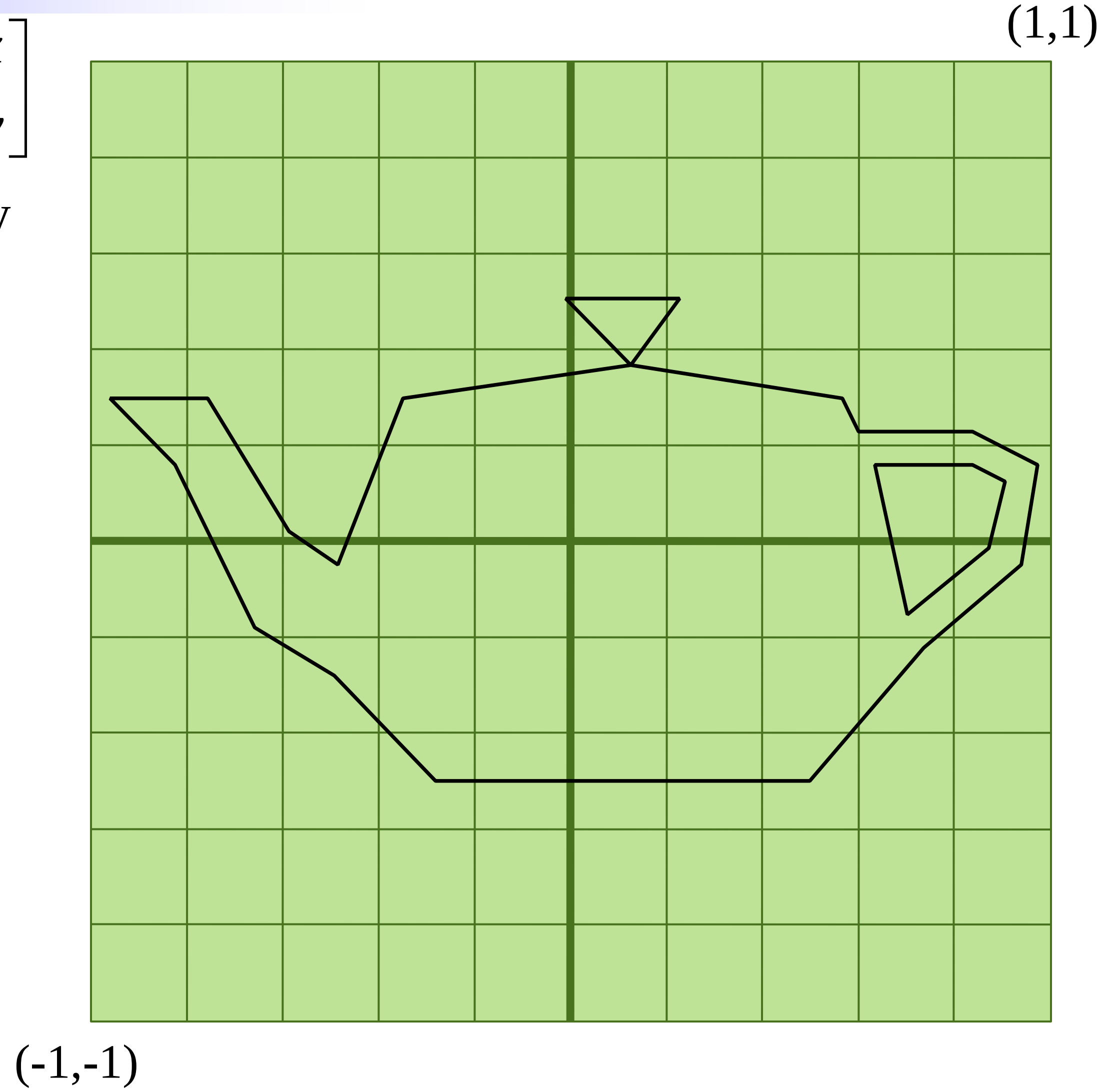
2-D Points

- Represents points and vertices as column vectors: $\begin{bmatrix} x \\ y \end{bmatrix}$



2-D Points

- Represents points and vertices as column vectors: $\begin{bmatrix} x \\ y \end{bmatrix}$
- Transform polygonal object by transforming its vertices



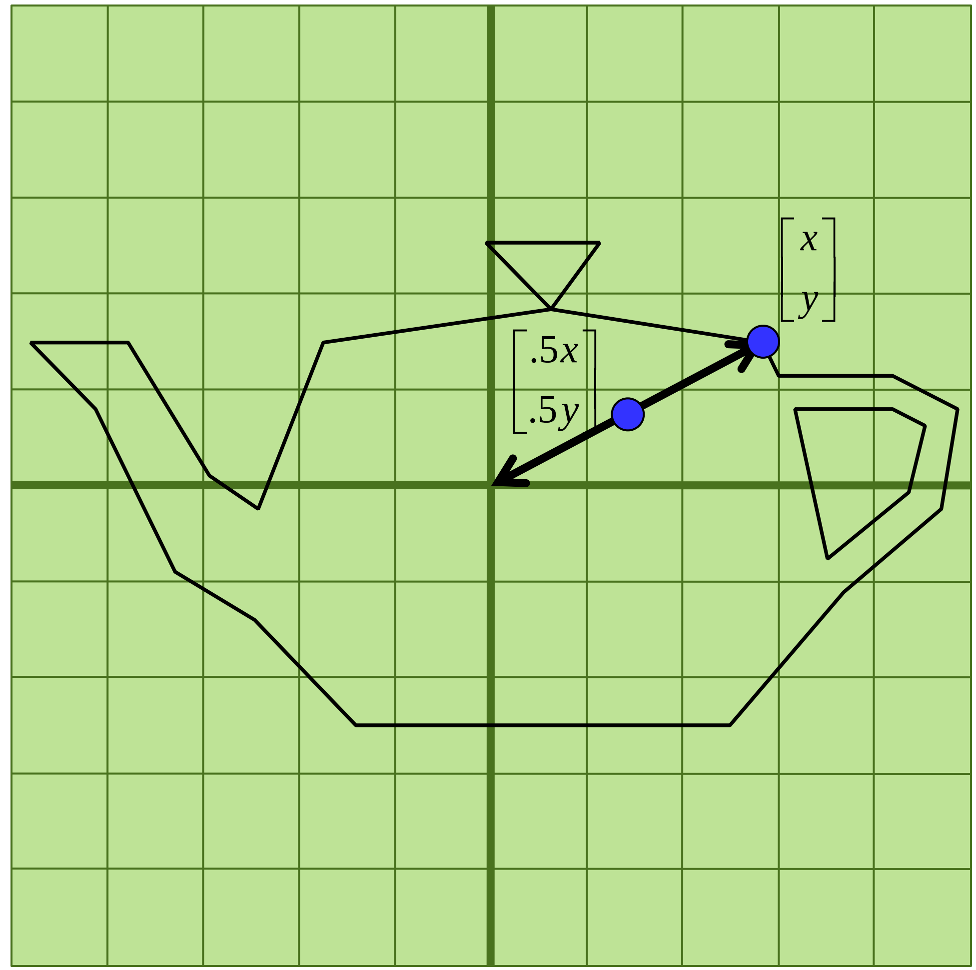
2-D Points

$$\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Represents points and vertices as column vectors: $\begin{bmatrix} x \\ y \end{bmatrix}$
- Transform polygonal object by transforming its vertices
- Scale by matrix multiplication

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix}$$

(1,1)



(-1,-1)

2-D Points

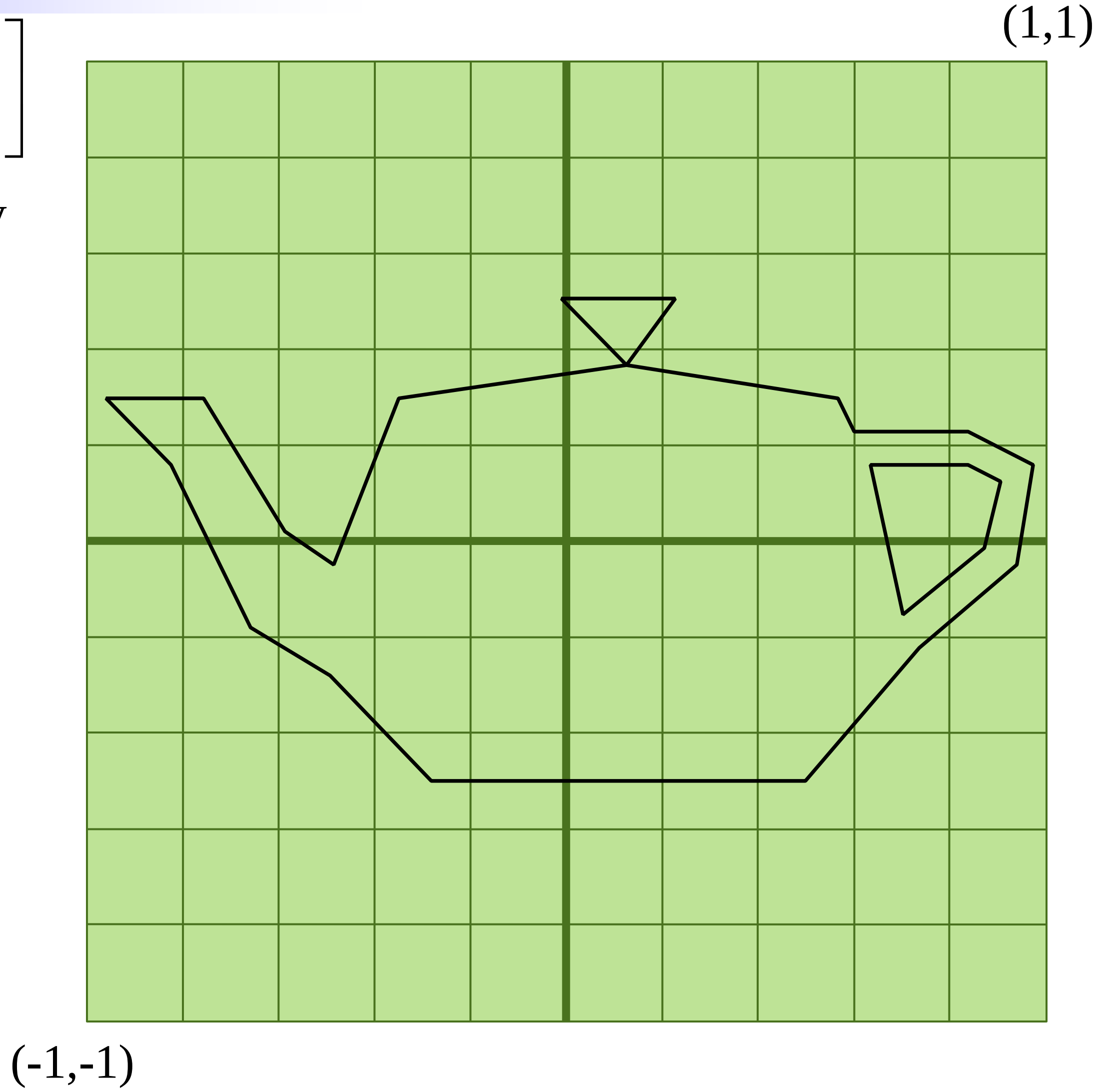
$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -.4 \end{bmatrix}$$

- Represents points and vertices as column vectors: $\begin{bmatrix} x \\ y \end{bmatrix}$
- Transform polygonal object by transforming its vertices
- Scale by matrix multiplication

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix}$$

- Translation via vector sum

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$



2-D Points

$$\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -.4 \end{bmatrix}\right)$$

$$\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -.4 \end{bmatrix}$$

- Represents points and vertices as column vectors: $\begin{bmatrix} x \\ y \end{bmatrix}$
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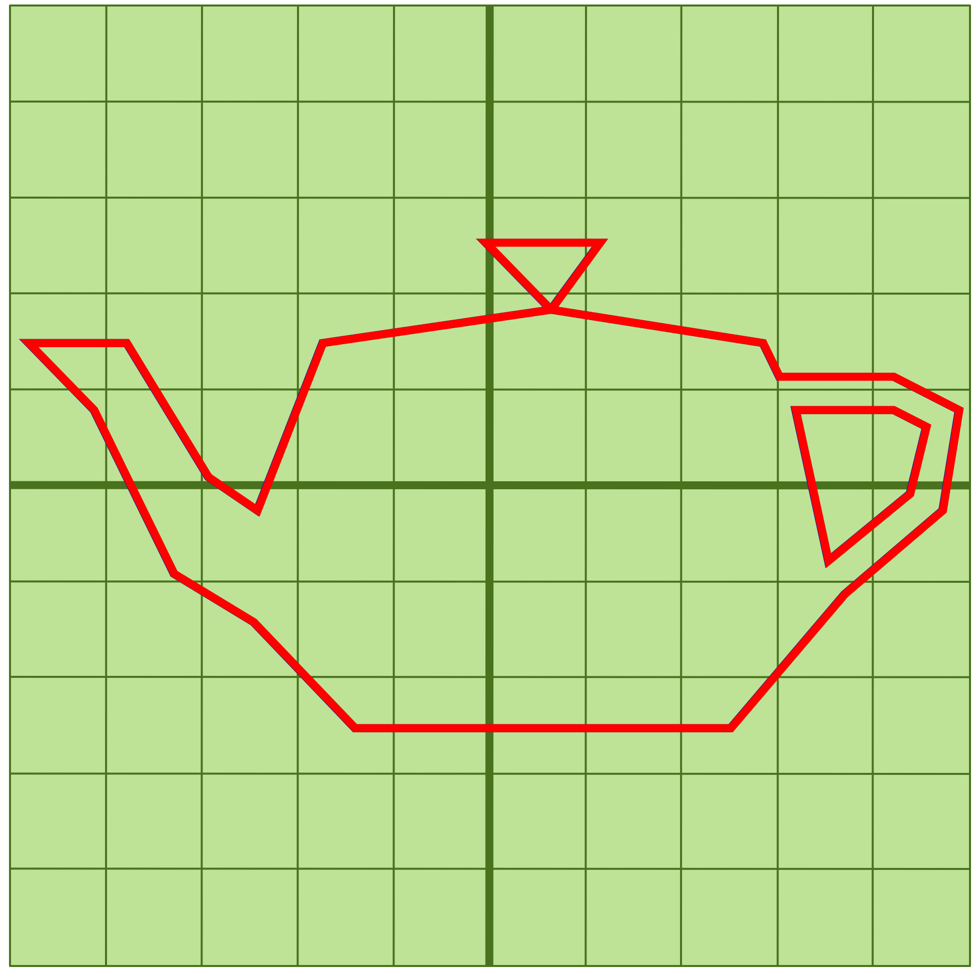
- Translation via vector sum

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

- Order is important
  - Translate then scale
  - Scale then translate

(1,1)



(-1,-1)

Homogeneous Coordinates

$$\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -.4 \end{bmatrix}\right)$$

- Translation by vector sum is cumbersome
- Add a extra coordinate
  - Called the homogeneous coordinate
  - For now, set to one
- Translation now expressed as a matrix
- Now we can compose scales and translations into a single matrix by matrix multiplication

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -.4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & -.4 \\ 0 & 0 & 1 \end{bmatrix}$$

Order Dependence

$$\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 0 \\ -.4 \end{bmatrix}\right)$$

$$\begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -.4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & -.2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 0 \\ -.4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -.4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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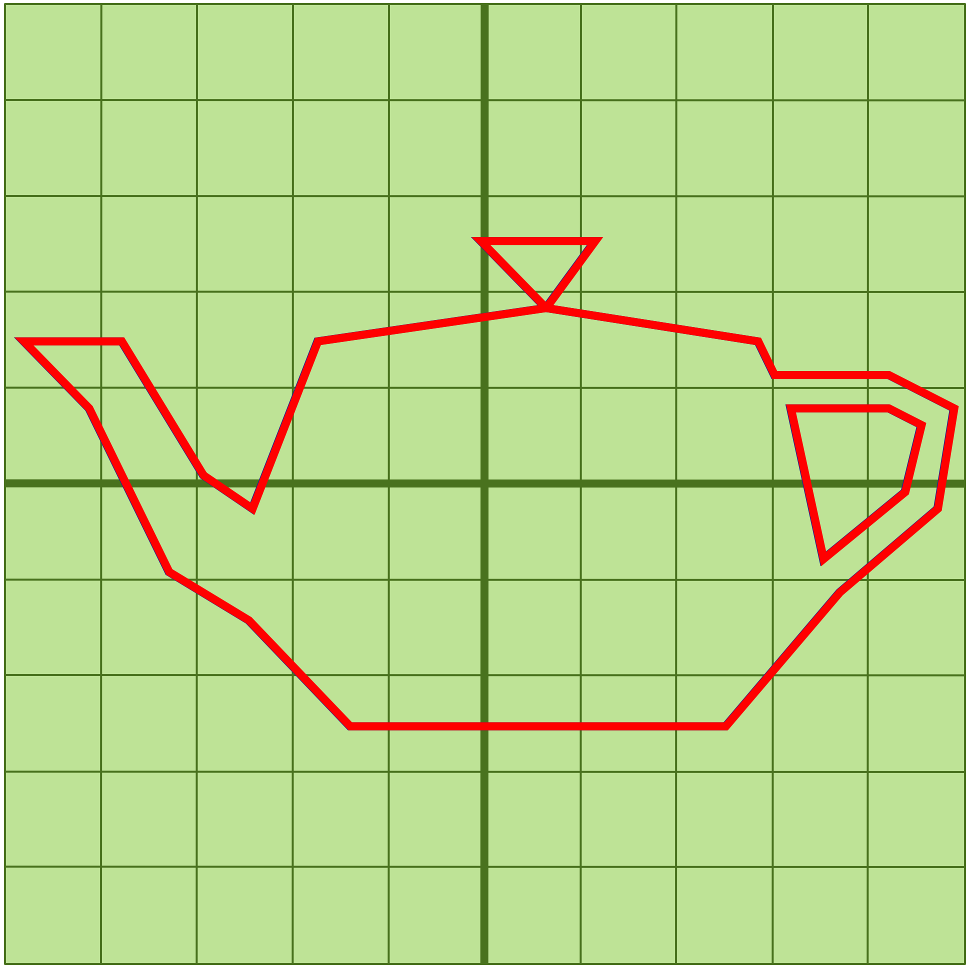


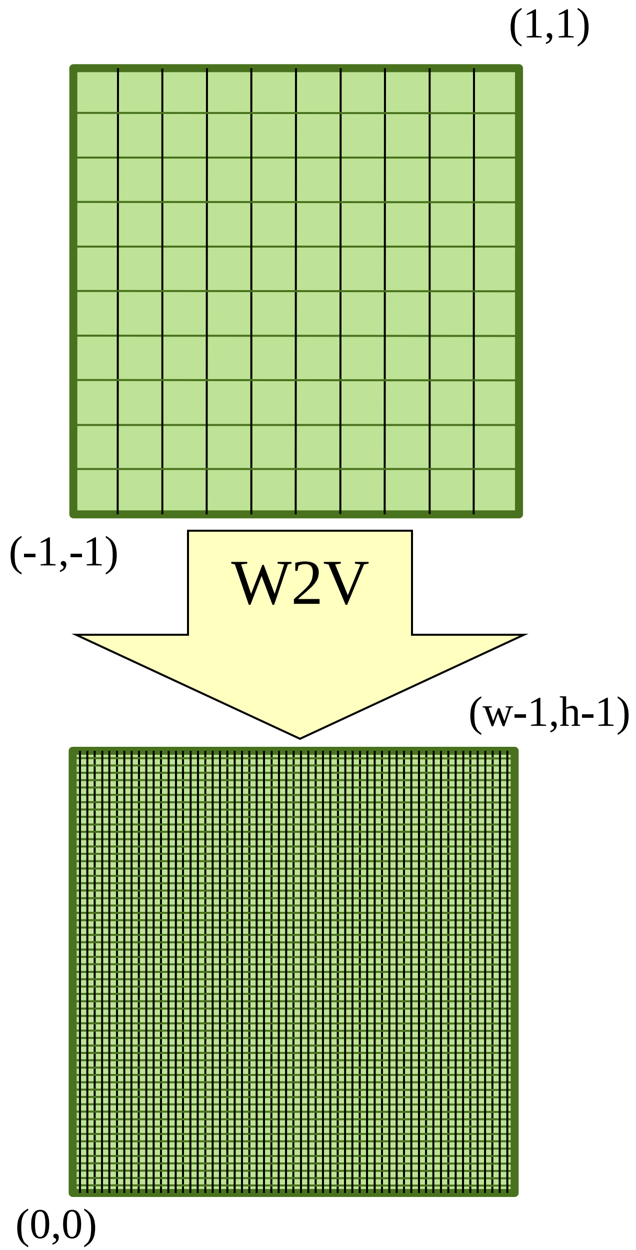
(-1,-1)

Window-to-Viewport

- First translate lower-left corner to (0,0) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- Then scale upper-right corner from (2,2) to (w-1,h-1)

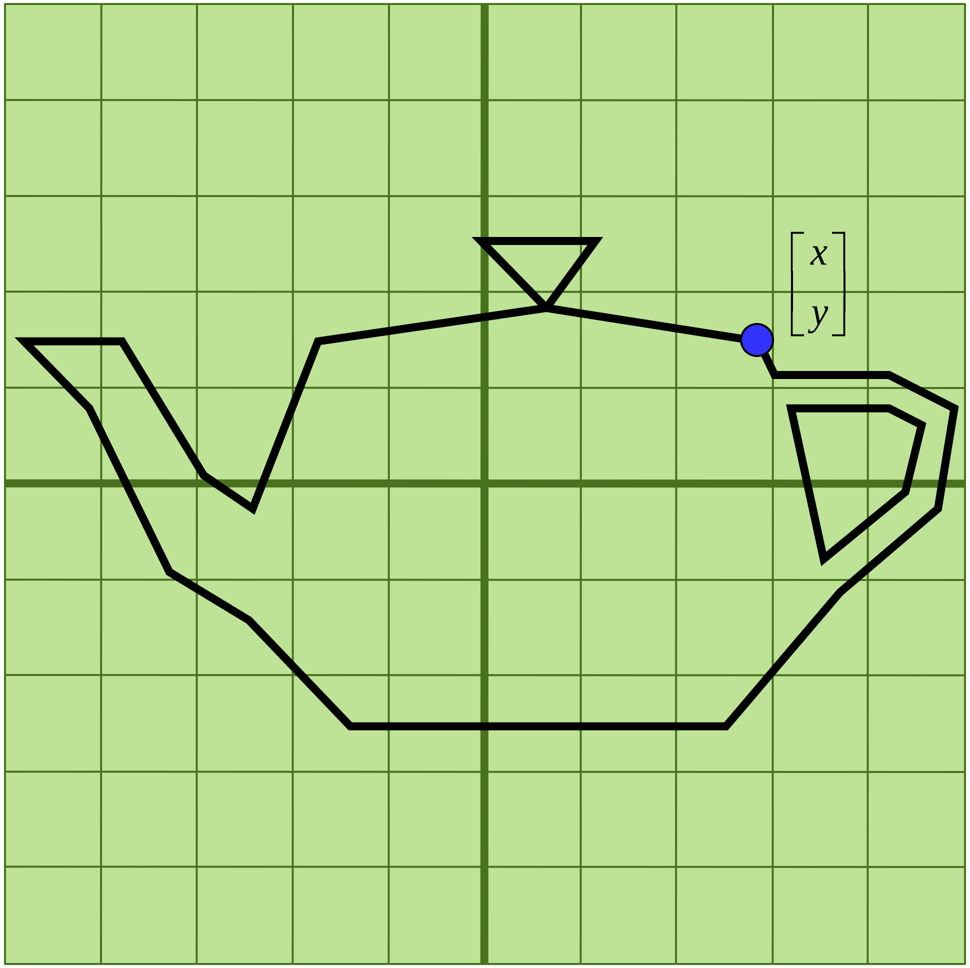
$$\begin{bmatrix} \frac{w-1}{2} & 0 & 0 \\ 0 & \frac{h-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- To get $\begin{bmatrix} \frac{w-1}{2} & 0 & \frac{w-1}{2} \\ 0 & \frac{h-1}{2} & \frac{h-1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$



2-D Rotation

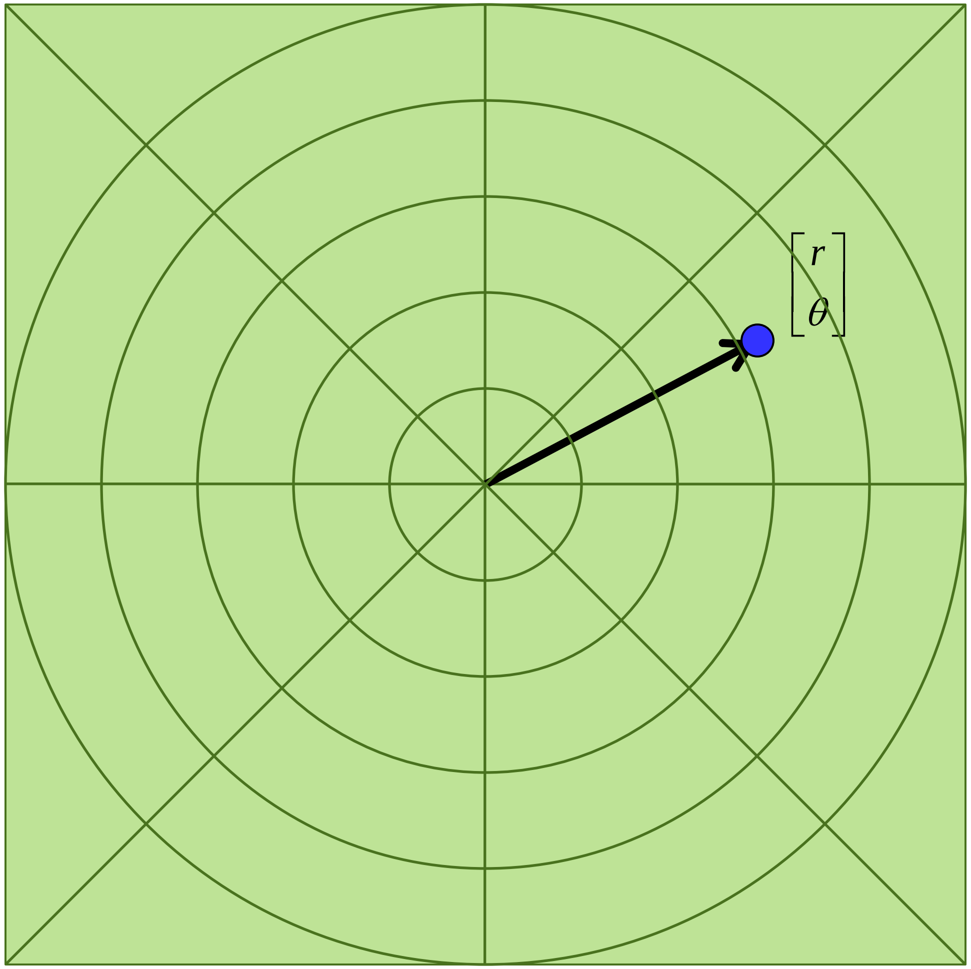
- Pick a point (x,y)



2-D Rotation

- Pick a point (x,y)
- Assume polar coords

  $x = r \cos\theta, y = r \sin\theta$

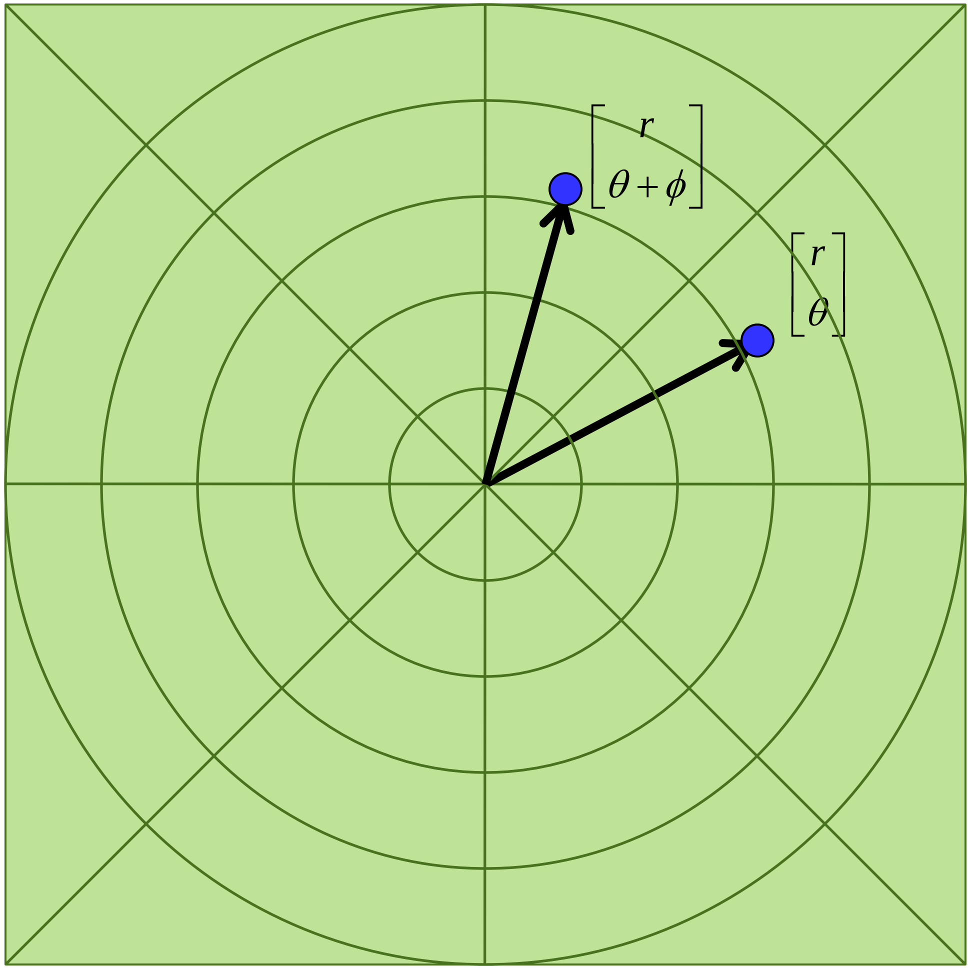


2-D Rotation

- Pick a point (x,y)
- Assume polar coords

  $x = r \cos \theta, y = r \sin \theta$
- Rotate about origin by ϕ

  $x' = r \cos \theta + \phi, y' = r \sin \theta + \phi$



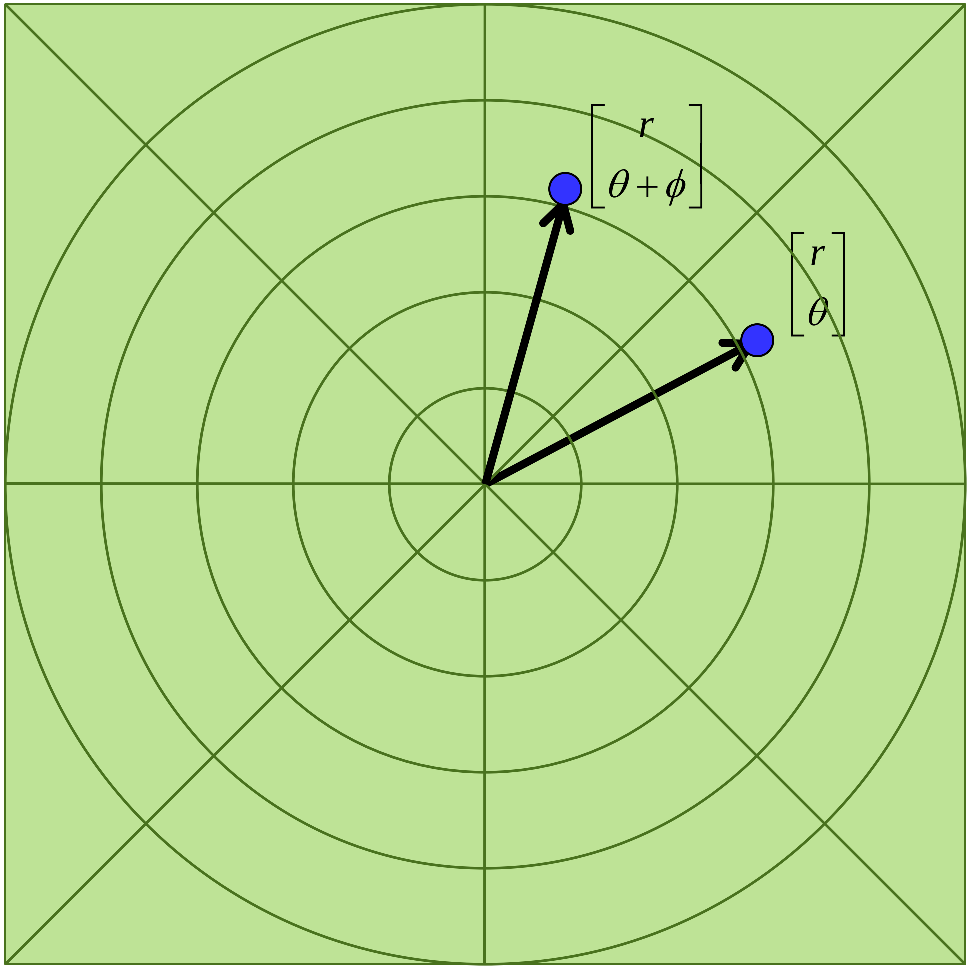
2-D Rotation

- Pick a point (x,y)
- Assume polar coords

  $x = r \cos \theta, y = r \sin \theta$
- Rotate about origin by ϕ

  $x' = r \cos \theta+\phi, y' = r \sin \theta+\phi$
- Recall trig. identities

  $x' = r (\cos\theta \cos \phi - \sin\theta \sin\phi)$
  $y' = r (\sin\theta \cos \phi + \cos\theta \sin\phi)$



2-D Rotation

- Pick a point (x,y)
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  $x = r \cos \theta, y = r \sin \theta$
- Rotate about origin by ϕ

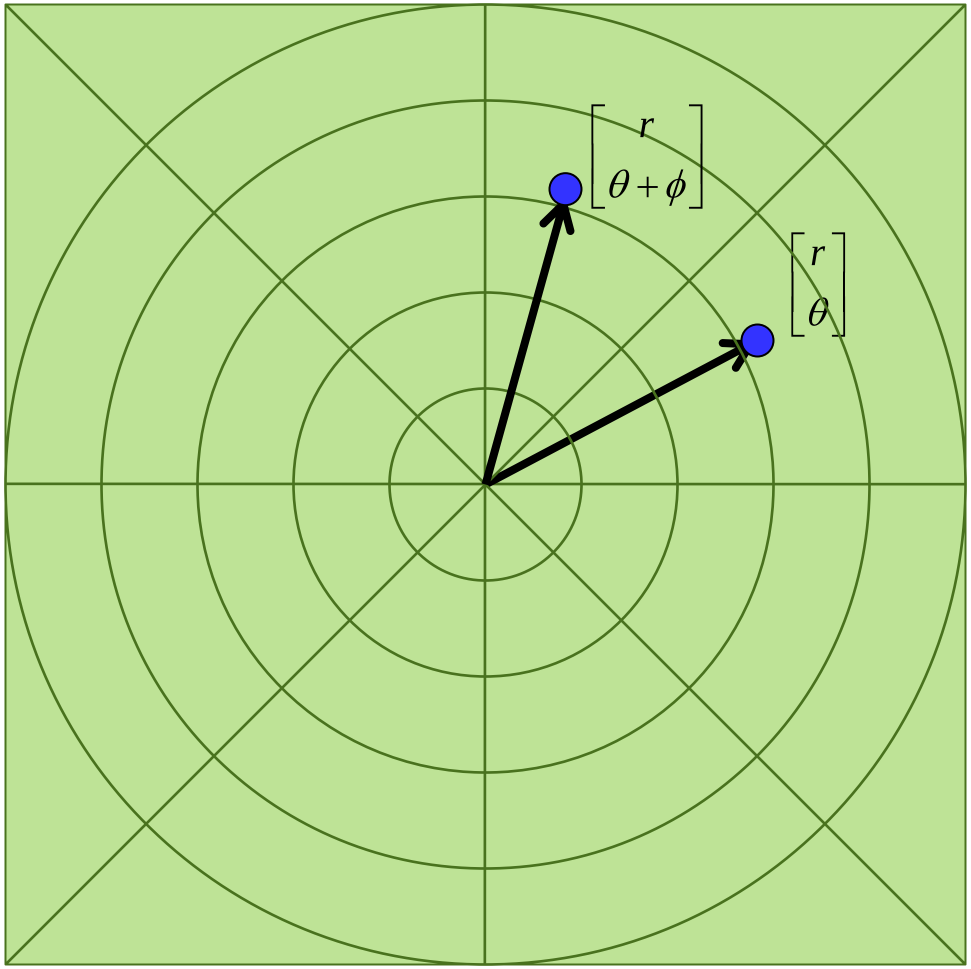
  $x' = r \cos \theta + \phi, y' = r \sin \theta + \phi$
- Recall trig. identities

  $x' = r (\cos\theta \cos \phi - \sin\theta \sin\phi)$

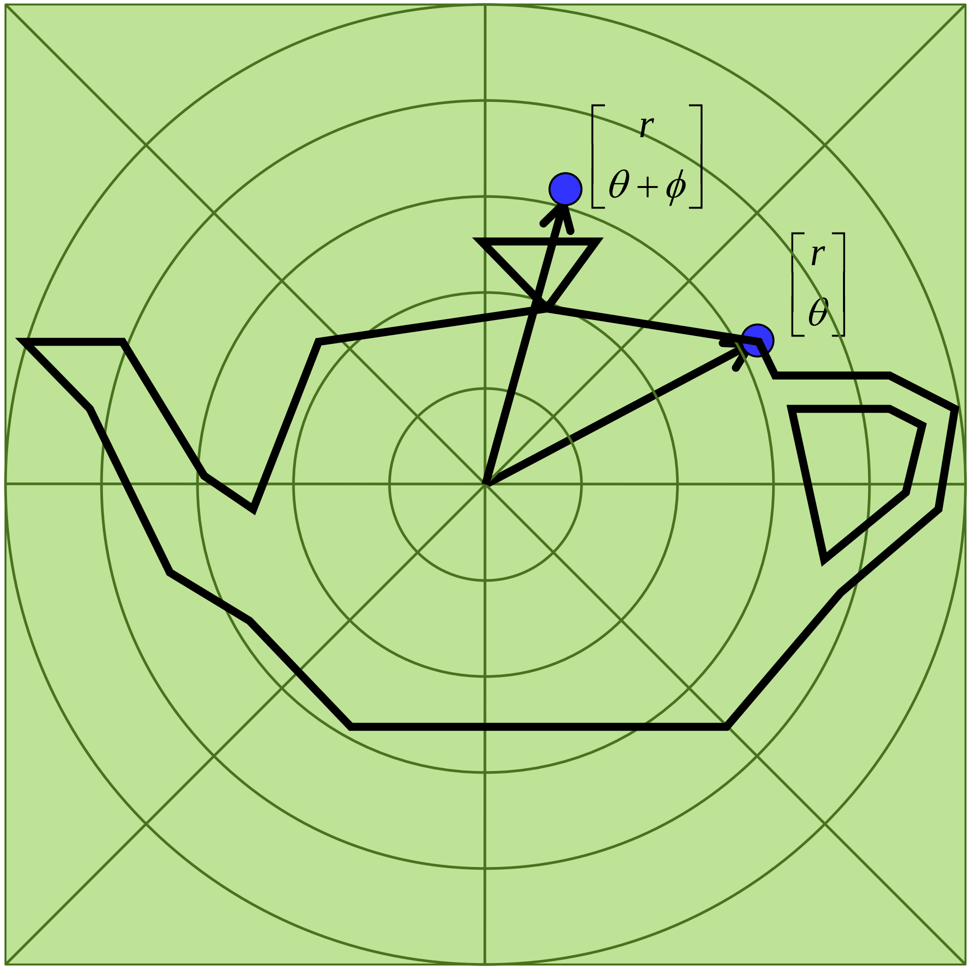
  $y' = r (\sin\theta \cos \phi + \cos\theta \sin\phi)$
- Rearrange terms

  $x' = \cos \phi (r \cos\theta) - \sin\phi (r \sin\theta)$

  $y' = (r \cos\theta) \sin\phi + (r \sin\theta) \cos \phi$



2-D Rotation



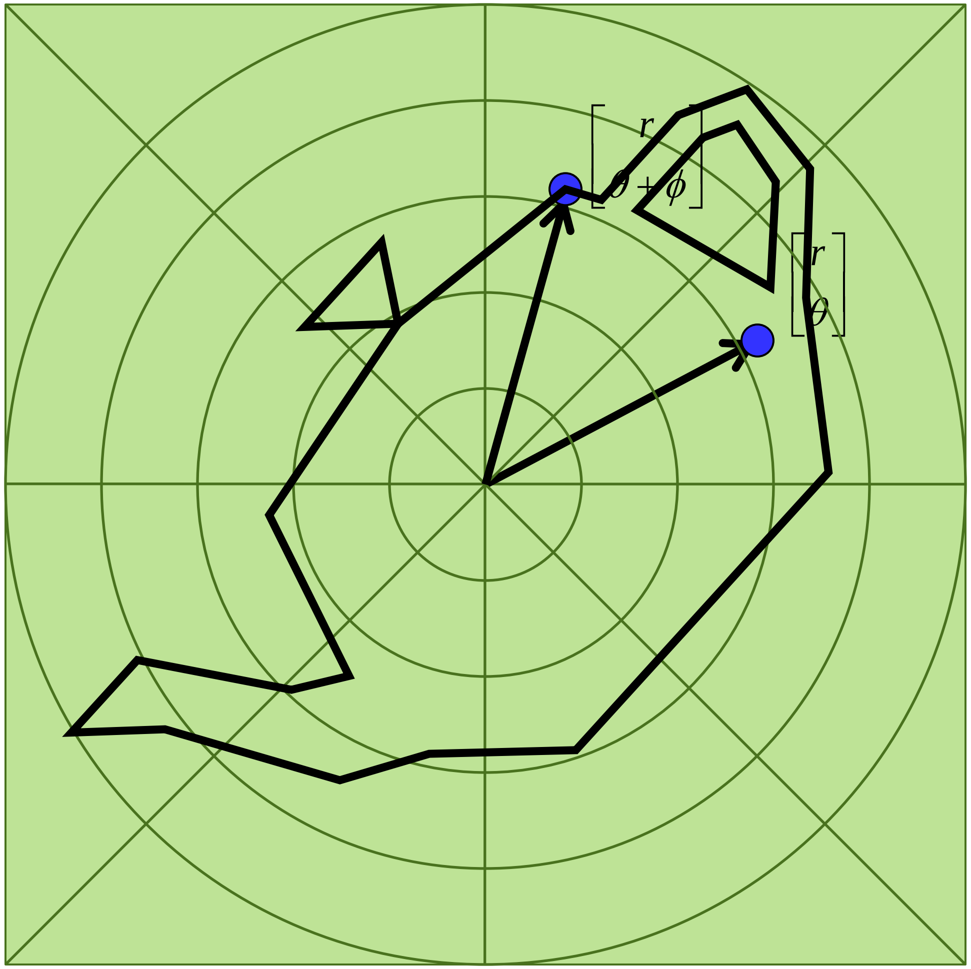
$$x' = \cos\phi\,(r\cos\theta) - \sin\phi\,(r\sin\theta)$$
$$y' = (r\cos\theta)\sin\phi + (r\sin\theta)\cos\phi$$

2-D Rotation

$$x' = \cos\phi\,(\underbrace{r\cos\theta}_{x}) - \sin\phi\,(\underbrace{r\sin\theta}_{y})$$

$$y' = (\underbrace{r\cos\theta}_{x})\sin\phi + (\underbrace{r\sin\theta}_{y})\cos\phi$$

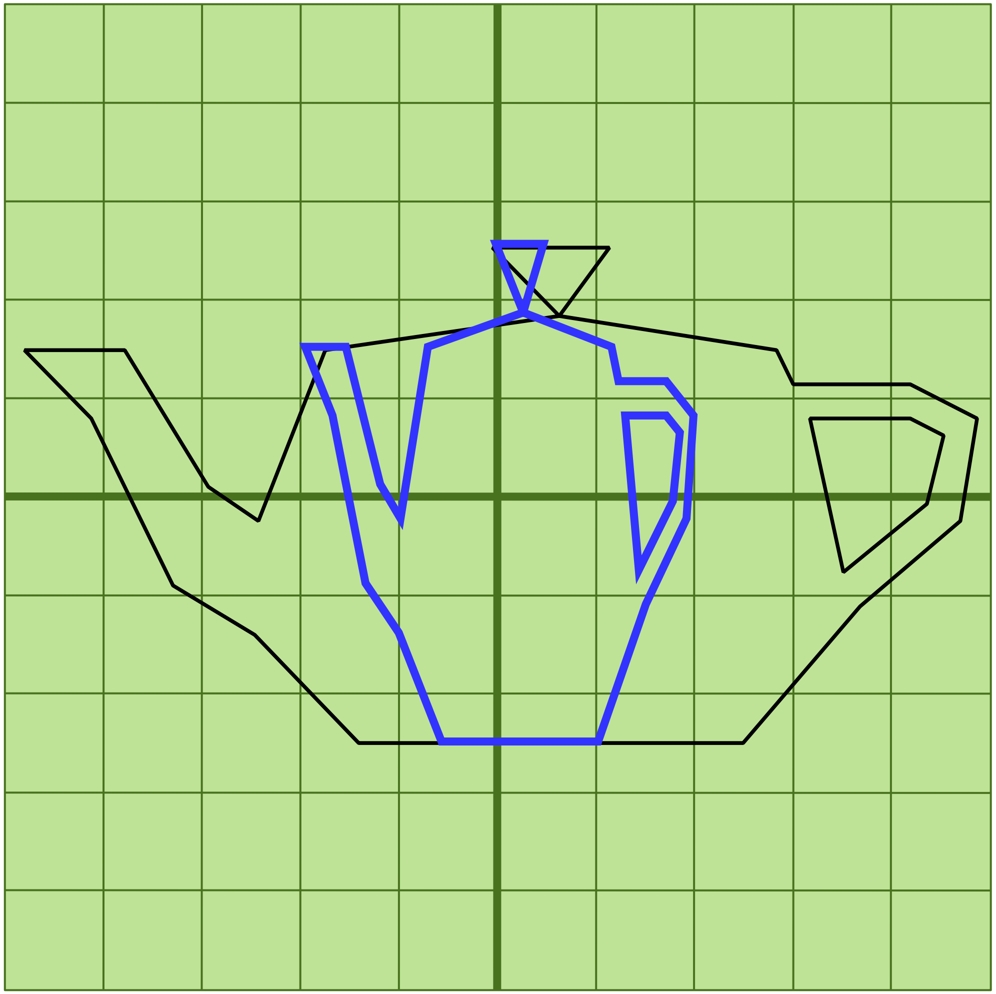
$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Squash

- Scale one coordinate by matrix multiplication

$$\begin{bmatrix} a & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ y \\ 1 \end{bmatrix}$$



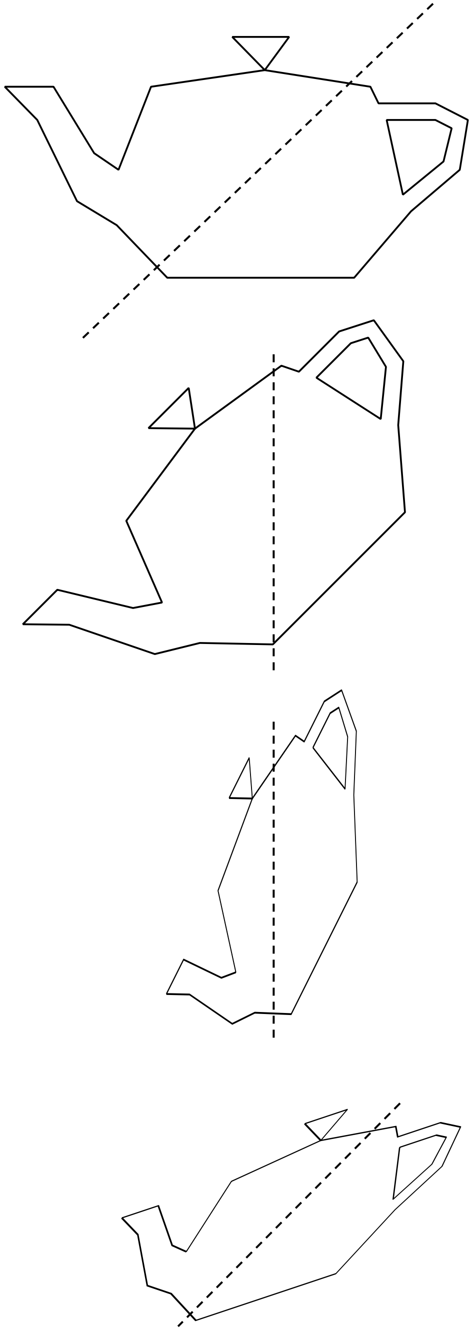
Squash

- Squash in arbitrary direction
  - Rotate direction into x
  - Squash in x
  - Rotate back

$$\begin{bmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \begin{bmatrix} a & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \begin{bmatrix} a\cos\theta & -a\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\theta + \sin^2\theta & (1-a)\cos\theta\sin\theta & \\ (1-a)\cos\theta\sin\theta & \cos^2\theta + a\sin^2\theta & \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Graphics Pipeline